TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 308, Number 1, July 1988

IWASAWA'S λ^--INVARIANT AND A SUPPLEMENTARY FACTOR IN AN ALGEBRAIC CLASS NUMBER FORMULA

KUNIAKI HORIE

ABSTRACT. Let l be a prime number and k an imaginary abelian field. Sinnott [**12**] has shown that the relative class number of k is expressed by the so-called index of the Stickelberger ideal of k, with a "supplementary factor" c^- in $\mathbf{N}/2 = \{n/2 | n \in \mathbf{N}\}$, and that if k varies through the layers of the basic $\mathbf{Z}_l$-extension over an imaginary abelian field, then c^- becomes eventually constant. On the other hand, c^- can take any value in $\mathbf{N}/2$ as k ranges over the imaginary abelian fields (cf. [**10**]). In this paper, we shall study relations between the supplementary factor c^- and Iwasawa's λ^--invariant for the basic $\mathbf{Z}_l$-extension over k, our discussion being based upon some formulas of Kida [**8**, **9**], those of Sinnott [**12**], and fundamental results concerning a finite abelian l-group acted on by a cyclic group. As a consequence, we shall see that the λ^--invariant goes to infinity whenever k ranges over a sequence of imaginary abelian fields such that the l-part of c^- goes to infinity.

Let $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote the field of rational numbers, the field of real numbers, and that of complex numbers, respectively. A finite abelian extension over $\mathbf{Q}$ contained in $\mathbf{C}$ will be called, simply, an abelian field. Let k be an imaginary abelian field, namely, an abelian field not contained in $\mathbf{R}$. This field k will be fixed throughout the following sections. We let h^- denote the ratio of the class number of k to that of $k^+ = k \cap \mathbf{R}$, which is known to be an integer. Let l be a fixed prime number and let $\mathbf{Z}_l$ denote the ring of l-adic integers. We write k_∞ for the basic $\mathbf{Z}_l$-extension over k in $\mathbf{C}$.

In this paper, we shall discuss relations between Iwasawa's λ^--invariant associated with k_∞ and the supplementary factor in an algebraic class number formula for h^- (due to Iwasawa and Sinnott), estimating the l-part of the latter by an elementary function of the former:

$$\begin{aligned} \operatorname{ord}_l(\varepsilon^- R : \varepsilon^- U) &\le \frac{l(\lambda^-(k) - 1)}{2l-3}\left(8^{(\lambda^-(k)-1)/(2l-3)} - 4\right) \quad \text{if } l > 2, \\ &\le \frac{64}{3}(5\lambda^-(k) + 4)(4^{\lambda^-(k)+1} - 1) \qquad \text{if } l = 2, \end{aligned}$$

so that

$$\lambda^-(k) \gg \log(\operatorname{ord}_l(\varepsilon^- R : \varepsilon^- U))$$

Received by the editors May 17, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11R20, 11R23; Secondary 11R18, 11R29.

Key words and phrases. Basic $\mathbf{Z}_l$-extension, Iwasawa's λ-invariant, abelian field, Stickelberger ideal, class number.

as k ranges over all imaginary abelian fields (for the above notations not defined, see the next section).

1. First we give some notations and review some fundamental results by Sinnott (for the details, see [**12**]).

Let K be any abelian extension over $\mathbb{Q}$ in $\mathbb{C}$, which is not necessarily of finite degree. We denote by $G(K/\mathbb{Q})$ the Galois group of K over $\mathbb{Q}$. For each positive integer r, let $T(r, K)$ and $Z(r, K)$ denote respectively the subgroups of $G(K/\mathbb{Q})$ generated by inertia groups and decomposition groups, for $K/\mathbb{Q}$, of all prime numbers dividing r. In particular, for any prime number p, $T(p, K)$ is the inertia group of p for $K/\mathbb{Q}$ and $Z(p, K)$ the decomposition group of p for $K/\mathbb{Q}$.

Next, assume further that $[K : \mathbb{Q}] < \infty$, so that K is an abelian field. Let $\mathfrak{S}(K)$ and $R(K)$ denote, respectively, the group rings of $G(K/\mathbb{Q})$ over $\mathbb{Q}$ and over the ring $\mathbb{Z}$ of rational integers:

$$R(K) = \mathbb{Z}[G(K/\mathbb{Q})] \subseteq \mathfrak{S}(K) = \mathbb{Q}[G(K/\mathbb{Q})].$$

It follows that $R(K)$ is a lattice in the $\mathbb{Q}$-vector space $\mathfrak{S}(K)$. At the same time, we can regard $\mathfrak{S}(K)$ as an $R(K)$-module in the obvious manner. For any subset H of $G(K/\mathbb{Q})$, let $|H|$ denote as usual the cardinality of H, and $s(H)$ the sum in $R(K)$ of all elements of H. For any prime number p, take a Frobenius automorphism τ of p for $K/\mathbb{Q}$. We then obtain the following element of $\mathfrak{S}(K)$;

$$(p, K)^* = \frac{1}{|T(p, K)|}\tau^{-1}s(T(p, K)),$$

which obviously does not depend on the choice of τ. Let n be any positive integer. We now define $U(n, K)$ to be the $R(K)$-module generated in $\mathfrak{S}(K)$ by

$$s(T(r, K)) \prod_{p|n/r} (1 - (p, K)^*)$$

as r ranges over all positive divisors of n, with p ranging over the prime divisors of n/r. It follows that $U(n, K)$ is a lattice of $\mathfrak{S}(K)$ and that, in particular, $U(1, K) = R(K)$.

In general, let X and Y be any lattices in a finite dimensional vector space V over $\mathbb{Q}$. Then there exists a $\mathbb{Q}$-linear automorphism Φ of V such that $\Phi(X) = Y$. We denote by $(X : Y)$ the absolute value of the determinant of Φ, which does not depend on the choice of Φ. Note that if $X \supseteq Y$, then $(X : Y)$ equals the index $[X : Y]$.

Let K be again any abelian field. Let α be any element in $\mathfrak{S}(K)$ and let n_1, n_2 be positive integers, so that both $\alpha U(n_1, K)$ and $\alpha U(n_2, K)$ are lattices in $\alpha\mathfrak{S}(K)$. It is known that if n_1 divides n_2, then $(\alpha U(n_1, K) : \alpha U(n_2, K))$ becomes a positive integer. Let f_K denote the conductor of K. It is also known that for every positive integer n, $U(n, K) = U((\overline{n}, f_K), K)$ where $\overline{n}$ denotes the product of distinct prime numbers dividing n. We put $U(K) = U(f_K, K)$.

Now, as to k, we let for simplicity

$$G = G(k/\mathbb{Q}), \quad \mathfrak{S} = \mathfrak{S}(k), \quad R = R(k), \quad \overline{\sigma}_p = (p, k)^*,$$
$$T_r = T(r, k), \quad Z_r = Z(r, k), \quad U_r = U(r, k), \quad f = f_k, \quad U = U(k) = U_{\overline{f}},$$

where p is any prime number and r any positive integer. Let j denote the complex conjugation of $\mathbb{C}$, so that the restriction $j|k$ is the complex conjugation of k. We define in $\mathfrak{S}$

$$\varepsilon^- = \tfrac{1}{2}(1 - j|k), \qquad \varepsilon^+ = \tfrac{1}{2}(1 + j|k).$$

These are idempotents of $\mathfrak{S}$. Let S be the Stickelberger ideal of k in the sense of Iwasawa-Sinnott, and let A be the ideal of R consisting of all $\alpha \in R$ such that $\varepsilon^+\alpha = as(G)/2$ for some $a \in \mathbb{Z}$. Then Theorem 2.1 of [**12**] states that S is a submodule of A with finite index and that

$$(1) \qquad [A : S] = \frac{(\varepsilon^- R : \varepsilon^- U)}{Q} h^-$$

where Q denotes the unit index of k. By Theorems 5.1, 5.3 of [**12**], if $\overline{f}$ is a prime number or G is a cyclic group, then

$$(\varepsilon^- R : \varepsilon^- U) = Q = 1 \quad \text{so that} \quad h^- = [A : S]$$

(see also [**3**]). In the case where k is a cyclotomic field, $(\varepsilon^- R : \varepsilon^- U)/Q$ is only a power of 2 and is calculated completely in [**11**]. Thus formula (1) may be viewed as an "algebraic" expression of h^- by $[A : S]$, with the "supplementary factor" $(\varepsilon^- R : \varepsilon^- U)/Q$. Since $Q = 1$ or 2, we are now interested in the value $(\varepsilon^- R : \varepsilon^- U)$.

We note here that, given any positive integer n, there exist infinitely many imaginary abelian fields $\mathfrak{k}$ such that $(\varepsilon_{\mathfrak{k}}^- R(\mathfrak{k}) : \varepsilon_{\mathfrak{k}}^- U(\mathfrak{k})) = n$ with $\varepsilon_{\mathfrak{k}}^- = \frac{1}{2}(1 - j|\mathfrak{k})$ in $\mathfrak{S}(\mathfrak{k})$. On the other hand, $(\varepsilon^- R : \varepsilon^- U)$ is a divisor of $([k : \mathbb{Q}]/2)^{[k:\mathbb{Q}]/2}$, and $\log[A : S] \sim \log h^-$ as k ranges over the imaginary abelian fields (cf. [**4, 10**]).

We shall use the following notations as well as the preceding ones. With K any abelian field as before, let K^+ denote the maximal real subfield of K: $K^+ = K \cap \mathbb{R}$. We then define

$$\lambda^-(K) = \lambda(K) - \lambda(K^+),$$

where $\lambda(K)$ and$\lambda(K^+)$ denote respectively the Iwasawa λ-invariants associated with the basic $\mathbb{Z}_l$-extensions over K and K^+. Obviously $\lambda^-(K) = 0$ if K is real. Next we define the character group associated with K. By a character of a finite abelian group, we mean a homomorphism of the group into the multiplicative group $\mathbb{C}^\times$ of $\mathbb{C}$. Each character ψ of $G(K/\mathbb{Q})$ can be extended to a $\mathbb{Q}$-algebra homomorphism $\mathfrak{S}(K) \to \mathbb{C}$ in the usual manner. Then ψ induces a primitive Dirichlet character ψ^* satisfying

$$\psi^*(p) = \psi((p, K)^*)$$

for every prime number p. Such a ψ^* will be called a character associated with K. Conversely, for any primitive Dirichlet character χ, there exist an abelian field K' and a character χ' of $G(K'/\mathbb{Q})$ such that

$$\chi(p) = \chi'((p, K')^*)$$

for all prime numbers p. The fixed field of $\mathrm{Ker}(\chi')$ will be denoted by $\mathbb{K}_\chi$, since it does not depend on the choice of K', χ'. Indeed $\mathbb{K}_\chi$ is the intersection of the abelian fields K' such that χ is a character associated with K'. Furthermore $\mathbb{K}_\chi$ is a cyclic extension over $\mathbb{Q}$ with degree and conductor equal to, respectively, the order and the conductor of χ. We denote by $\mathfrak{X}_K$ the set of all characters associated with K. If K is imaginary, then we denote by $\mathfrak{X}_K^-$ the set of odd Dirichlet characters in $\mathfrak{X}_K$:

$$\mathfrak{X}_K^- = \{\chi \in \mathfrak{X}_K;\ \chi(-1) = -1\}.$$

For each prime number p, the notation $[X : Y]_p$ means the highest power of p dividing $[X : Y]$, where Y is a subgroup of a group X with finite index or X is a finite extension over a field Y.

2. Let v be a prime number and let ord_v denote the v-adic exponential valuation. Let k' denote the maximal subfield in k of degree a power of v and f' the product of prime numbers ramified in k'. We write μ for the Möbius function.

LEMMA 1. *If v is odd, then*

$$\mathrm{ord}_v(\varepsilon^- R : \varepsilon^- U) = \frac{1}{2}\sum_t \left(\sum_{\substack{r \\ j|k \notin Z_r}} \mu\left(\frac{r}{t}\right) \frac{[G : Z_r]}{[G : Z_r]_v} \right) \mathrm{ord}_v(R(k') : U(t, k')),$$

where t ranges over the positive integers dividing f' with r ranging over all positive integers such that $t|r$, $r|f'$, and $j|k \notin Z_r$. For the case $v = 2$,

$$\mathrm{ord}_2(\varepsilon^- R : \varepsilon^- U) = \sum_t \left(\sum_r \mu\left(\frac{r}{t}\right) \frac{[G : Z_r]}{[G : Z_r]_2} \right) \mathrm{ord}_2(\varepsilon_{k'}^- R(k') : \varepsilon_{k'}^- U(t, k')).$$

Here $\varepsilon_{k'}^- = \frac{1}{2}(1 - j|k') \in \mathfrak{S}(k')$, and t ranges over the positive integers dividing f' with r ranging over all positive divisors of f' divisible by t.

PROOF. Let k'' be the maximal subfield in k of degree prime to v: $k = k'k''$, $k' \cap k'' = \mathbf{Q}$. For each character χ in $\mathfrak{X}_{k''}$, let f'_χ denote the product of prime numbers p dividing f' and satisfying $\chi(p) = 1$. Theorem 5.2 of [**12**] then implies that

$$\begin{aligned} (2) \qquad v^{\mathrm{ord}_v(\varepsilon^- R : \varepsilon^- U)} &= \prod_{\chi \in \mathfrak{X}_{k''}^-} (R(k') : U(f'_\chi, k')), && \text{if } v > 2, \\ &= \prod_{\chi \in \mathfrak{X}_{k''}} (\varepsilon_{k'}^- R(k') : \varepsilon_{k'}^- U(f'_\chi, k')), && \text{if } v = 2. \end{aligned}$$

Let t be a positive integer dividing f'. In the case $v > 2$, the number of characters χ in $\mathfrak{X}_{k''}^-$ with $f'_\chi = t$ is equal to the sum

$$\frac{1}{2}\sum_r \mu\left(\frac{r}{t}\right) [G(k''/\mathbf{Q}) : Z(r, k'')],$$

which is taken over the positive integers r such that $t|r$, $r|f'$, and $j|k \notin Z_r$. Similarly, in the case $v = 2$, the number of χ in $\mathfrak{X}_{k''}$ with $f'_\chi = t$ is equal to the sum

$$\sum_r \mu\left(\frac{r}{t}\right) [G(k''/\mathbf{Q}) : Z(r, k'')],$$

which is taken over the positive divisors r of f' divisible by t. Hence, applying ord_v to (2), we obtain the lemma.

For each integer $n \geq 0$, let k_n denote the intermediate field of k_∞/k with degree l^n over k and let, in $\mathfrak{S}(k_n)$,

$$\varepsilon_n^- = \tfrac{1}{2}(1 - j|k_n).$$

It is known that if n is sufficiently large, then $(\varepsilon_n^- R(k_n) : \varepsilon_n^- U(k_n))$ becomes a positive integer independent of n (cf. Theorem 6.1 of [**12**]). We can therefore define a nonnegative integer

$$a_v = \lim_{n\to\infty} \mathrm{ord}_v(\varepsilon_n^- R(k_n) : \varepsilon_n^- U(k_n)).$$

Let f_∞ denote the product of prime numbers ramified in k_∞; $f_\infty = \overline{fl}$. Let r be any positive integer dividing f_∞. Clearly

$$\lim_{n\to\infty} [G(k_n/\mathbf{Q}) : Z(r, k_n)]_v = [G(k_\infty/\mathbf{Q}) : Z(r, k_\infty)]_v,$$

and we have

$$[G(k_\infty/\mathbf{Q}) : Z(r, k_\infty)]_v = [G : Z_r]_v, \quad \text{unless } v = l. \tag{3}$$

Now, with p varying through the prime divisors of r, let

$$u(r) = \max_p u(p),$$

where

$$u(p) = \mathrm{ord}_l |T_l| \quad \text{or} \quad \max\left(\mathrm{ord}_l [G : T_p], \mathrm{ord}_l \frac{p^{2(l-1)} - 1}{4l}\right)$$

according as $p = l$ or $p \neq l$. Then the primes of $k_{u(r)}$ dividing r are undecomposed in k_∞, so that

$$[G(k_\infty/\mathbf{Q}) : Z(r, k_\infty)]_l = [G(k_{u(r)}/\mathbf{Q}) : Z(r, k_{u(r)})]_l. \tag{4}$$

We next put, for each positive integer t dividing f_∞,

$$\begin{aligned} a_v(t) &= \frac{1}{2} \sum_{\substack{r \\ j|k \notin Z_r}} \mu\left(\frac{r}{t}\right) \frac{[G(k_\infty/\mathbf{Q}) : Z(r, k_\infty)]}{[G(k_\infty/\mathbf{Q}) : Z(r, k_\infty)]_v}, && \text{if } v > 2, \\ &= \sum_r \mu\left(\frac{r}{t}\right) \frac{[G(k_\infty/\mathbf{Q}) : Z(r, k_\infty)]}{[G(k_\infty/\mathbf{Q}) : Z(r, k_\infty)]_2}, && \text{if } v = 2. \end{aligned}$$

Here r ranges over the positive divisors of f_∞ divisible by t, with $j|k \notin Z_r$ if $v > 2$. Note that

$$\frac{[G(k_\infty/\mathbf{Q}) : Z(r, k_\infty)]}{[G(k_\infty/\mathbf{Q}) : Z(r, k_\infty)]_v} = \prod_w [G(k_\infty/\mathbf{Q}) : Z(r, k_\infty)]_w,$$

the product taken over the prime divisors w of $l[k : \mathbf{Q}]$ different from v. Thus, in the case $v \neq l$, we obtain from Lemma 1 the following proposition, which, together with (3) and (4), gives a somewhat explicit expression of a_v.

PROPOSITION 1. *Assume that $v \neq l$. Let $\varepsilon' = 1$ or $\varepsilon_{k'}^-$ in $\mathfrak{S}(k')$ according as $v > 2$ or $v = 2$. Then*

$$a_v = \sum_t a_v(t) \mathrm{ord}_v(\varepsilon' R(k') : \varepsilon' U(t, k')),$$

the sum taken over the positive integers t dividing f_∞.

By this result and by the proof of Lemma 1 in [**4**] or the remark to Proposition 1 in [**10**], one can find an upper bound for a_v, $v \neq l$, depending only on k (and l). Our main interest, however, lies in the case $v = l$.

3. From now on, we shall only consider the case where the prime v in the preceding section coincides with l : $v = l$. Let k' denote as before the maximal subfield in k of degree a power of l, and k'' the maixmal subfield in k of degree prime to l. Let us start with showing some lemmas.

As is well known, a finitely generated module over a noetherian ring is a noetherian module. Moreover a usual proof of this fact provides us with the following (see, e.g., [**13**]):

LEMMA 2. *Let n and u be positive integers, N a noetherian ring such that any ideal of N is generated by at most n elements, and M an N-module generated by u elements. Then any N-submodule of M is generated by at most nu elements.*

LEMMA 3. *Let n be a positive integer and T a finite cyclic group. Then any ideal of the group ring $(\mathbb{Z}/l^n\mathbb{Z})[T]$ is generated by at most n elements.*

PROOF. We prove this by mathematical induction on n as follows. It is well known that $(\mathbb{Z}/l\mathbb{Z})[T]$ is a principal ideal ring (but is not an integral domain unless $T = \{1\}$). Hence the lemma holds when $n = 1$. Now, with assuming that the lemma holds when $n = \nu \geq 1$, let $B = (\mathbb{Z}/l^{\nu+1}\mathbb{Z})[T]$ and let $\mathfrak{A}$ be any ideal of B. Since lB is isomorphic to $(\mathbb{Z}/l^{\nu}\mathbb{Z})[T]$ as a B-module, the above assumption implies that the B-submodule $\mathfrak{A} \cap lB$ of lB is generated by at most ν elements. Furthermore the ring B/lB is isomorphic to $(\mathbb{Z}/l\mathbb{Z})[T]$, so that the ideal $(\mathfrak{A} + lB)/lB$ of B/lB is principal. The canonical B-isomorphism $(\mathfrak{A} + lB)/lB \cong \mathfrak{A}/(\mathfrak{A} \cap lB)$ then shows that $\mathfrak{A}$ itself is generated by at most $\nu + 1$ elements over B. This is to be proved.

The following lemma is an immediate consequence of Lemmas 2 and 3.

LEMMA 4. *Let n, u be positive integers, T a finite cyclic group, and M a module over $(\mathbb{Z}/l^n\mathbb{Z})[T]$ generated by u elements. Then any $(\mathbb{Z}/l^n\mathbb{Z})[T]$-submodule of M is generated by at most nu elements.*

For each positive integer u dividing $\overline{f}$, we put

$$\overline{\sigma}_u = \prod_{p|u} \overline{\sigma}_p$$

in $\mathfrak{S}$, where p ranges over the prime numbers dividing u. Let I be any subgroup of G. For any R-module D, we let D^I denote the submodule of D consisting of all elements in D fixed by I:

$$D^I = \{x \in D; \tau x = x \text{ for every } \tau \in I\}.$$

The notation $|I|_l$ denotes the highest power of l dividing the cardinality $|I|$.

LEMMA 5. *Let K be any abelian field, r a positive integer dividing $\overline{f_K}$, and p a prime number dividing $\overline{f_K}/r$. Then*

(i) $\mathrm{ord}_l(U(r,K) : U(rp,K)) \leq 2[G(K/\mathbb{Q}) : T(p,K)]\mathrm{ord}_l|T(r,K)|$,

(ii) *for any α in $\mathfrak{S}(K)$ and any cyclic subgroup H of $Z(p,K)$,*

$$\mathrm{ord}_l(\alpha U(r,K) : \alpha U(rp,K)) \leq 2^{\rho}[G(K/\mathbb{Q}) : H](\mathrm{ord}_l|T(p,K)|)^2$$

where ρ is the number of prime divisors of r.

PROOF. It suffices to show the lemma for the case $K = k$, without using the assumption that k is imaginary.

To prove (i), we note that $|T_n|\overline{\sigma}_n \in s(T_n)R$ for every positive integer n dividing $\overline{f}$. Hence

$$|T_r| = |T_r| \sum_{u|r} \left(\overline{\sigma}_u \prod_{w|r/u} (1 - \overline{\sigma}_w) \right) \in U_r,$$

where the sum is taken over the positive integers u dividing r, with w ranging over the prime divisors of r/u. Since $|T_r|U_r \subseteq R$, it follows that

$$|T_r|R \subseteq U_r \subseteq R/|T_r|.$$

This implies

$$|T_r|R^{T_p} = |T_r|s(T_p)R \subseteq s(T_p)U_r \subseteq U_r^{T_p} \subseteq R^{T_p}/|T_r|.$$

On the other hand, by [**12**],

$$s(T_p)U_r \subseteq U_{rp}^{T_p} \subseteq U_r^{T_p}, \qquad (U_r : U_{rp}) = [U_r^{T_p} : U_{rp}^{T_p}].$$

Therefore $(U_r : U_{rp})$ is a divisor of the index

$$[R^{T_p}/|T_r| : |T_r|R^{T_p}] = |T_r|^{2[G : T_p]}$$

and, in particular, we have

$$\mathrm{ord}_l(U_r : U_{rp}) \leq 2[G : T_p]\mathrm{ord}_l|T_r|.$$

Let us next prove (ii) for the special case where $\alpha = 1$ (and $K = k$); the general case can be shown quite similarly. Now, let H be any cyclic subgroup of Z_p and let

$$\varepsilon_p = s(T_p)/|T_p|.$$

We then see that

$$\varepsilon_p U_r \supseteq U_r^{T_p} \supseteq U_{rp}^{T_p} \supseteq s(T_p)U_r = |T_p|\varepsilon_p U_r.$$

Let X and Y be the l-primary parts of the finite abelian groups $\varepsilon_p U_r/U_{rp}^{T_p}$ and $U_r^{T_p}/U_{rp}^{T_p}$, respectively. It also follows from the definition of U_r that $\varepsilon_p U_r$, as well as U_r, is generated by 2^ρ elements over R, so that $\varepsilon_p U_r$ has at most $2^\rho[G : H]$ generators as a $\mathbb{Z}[H]$-module. Hence X becomes a $(\mathbb{Z}/|T_p|_l\mathbb{Z})[H]$-module generated by at most $2^\rho[G : H]$ elements. Obviously, Y is then a submodule of X. It therefore follows from Lemma 4 that Y has at most $2^\rho[G : H]\mathrm{ord}_l|T_p|$ generators over $(\mathbb{Z}/|T_p|_l\mathbb{Z})[H]$. Furthermore, since $(1 - \overline{\sigma}_p)U_r^{T_p} \subseteq U_{rp}^{T_p}$, we have $Y^{Z_p} = Y$, which shows that Y is viewed as a $\mathbb{Z}/|T_p|_l\mathbb{Z}$-module of rank $\leq 2^\rho[G : H]\mathrm{ord}_l|T_p|$. Consequently

$$|Y| \leq |T_p|_l^{2^\rho[G:H]\mathrm{ord}_l|T_p|}.$$

As $|Y|$ is the highest power of l dividing $(U_r : U_{rp})$, this means the inequality in (ii) for the case where $\alpha = 1$ and $K = k$.

By means of the above lemma and some results in [**12**], we shall prove the following two propositions.

PROPOSITION 2. *Suppose that l is odd. Let P be the set of prime numbers ramified in k' whose prime divisors of k^+ are decomposed in k, and let m be the product of all primes in P. If $(\varepsilon^- R\colon \varepsilon^- U)$ is divisible by l, then $|P| \geq 3$ and*

$$\begin{aligned}\operatorname{ord}_l(\varepsilon^- R : \varepsilon^- U) \\ &\leq \frac{1}{|P|}\left((2^{|P|-1}-2)\sum_{p\in P\setminus\{l\}}(\operatorname{ord}_l|T_p|)^2[G : Z_p][Z_p : T_l\cap Z_p]_l \right.\\ &\qquad\left. + \kappa(|P|-2)[G : Z_l][Z_l : T_l]_l\right).\end{aligned}$$

Here $\kappa = \operatorname{ord}_l|T_{m/l}|$ or 0 according as l lies in P or not.

PROOF. We assume that $l|(\varepsilon^- R : \varepsilon^- U)$. Let f' be, as before, the product of prime numbers ramified in k'. For any $\chi \in \mathfrak{X}^-_{k''}$ and for any positive integer t dividing f', let t_χ denote the product of prime divisors p of t with $\chi(p) = 1$. It then follows that

$$f'_\chi = m_\chi, \quad \text{for } \chi \in \mathfrak{X}^-_{k''}.$$

Since l is odd, this and Theorem 5.2 of [**12**] show that

$$\operatorname{ord}_l(\varepsilon^- R : \varepsilon^- U) = \sum_{\chi\in\mathfrak{X}^-_{k''}} \operatorname{ord}_l(R(k') : U(f'_\chi, k')) = \operatorname{ord}_l(\varepsilon^- R : \varepsilon^- U_m) \tag{5}$$

and, in particular, that $l|(\varepsilon^- R : \varepsilon^- U_m)$. However, if $|P| \leq 2$, then $(\varepsilon^- R : \varepsilon^- U_m)$ is a power of 2 by Proposition 5.2 of [**12**]. Thus we must have $|P| \geq 3$.

Now, for any nonnegative integers n and u with $n \geq u$, we define as usual

$${}_nC_u = n!/u!(n-u)!.$$

We next let

$$M_i = \frac{1}{|P|\,{}_{|P|-1}C_{i-1}}\sum_{r,p}\operatorname{ord}_l(\varepsilon^- U_r : \varepsilon^- U_{rp}), \quad \text{for } i = 1, \dots, |P|, \tag{6}$$

where r ranges over the positive integers dividing m and divisible by exactly $i-1$ prime numbers, with p ranging over all prime divisors of m/r. It is easy to see that

$$\operatorname{ord}_l(\varepsilon^- R : \varepsilon^- U_m) = \sum_{i=1}^{|P|} M_i,$$

because

$$(\varepsilon^- R : \varepsilon^- U_m) = \prod_{i=1}^{|P|}(\varepsilon^- U_{q_1\cdots q_{i-1}} : \varepsilon^- U_{q_1\cdots q_i})$$

for any permutation $(q_1, \dots, q_{|P|})$ of the prime numbers dividing m. Furthermore, as mentioned above, $M_1 = M_2 = 0$ by Proposition 5.2 of [**12**]. Hence it follows from (5) that

$$\operatorname{ord}_l(\varepsilon^- R : \varepsilon^- U) = \sum_{i=3}^{|P|} M_i. \tag{7}$$

To estimate M_i, $3 \le i \le |P|$, let us now look at $\mathrm{ord}_l(\varepsilon^- U_r : \varepsilon^- U_{rp})$ in (6). Again by Theorem 5.2 of [**12**],

$$\begin{aligned}\mathrm{ord}_l(\varepsilon^- U_r : \varepsilon^- U_{rp}) &= \sum_{\chi \in \mathfrak{X}^-_{k''}} \mathrm{ord}_l(U(r_\chi, k') : U((rp)_\chi, k')) \\ &= \sum_{\substack{\chi \in \mathfrak{X}^-_{k''} \\ \chi(p)=1}} \mathrm{ord}_l(U(r_\chi, k') : U(r_\chi p, k')).\end{aligned}$$

Here, noting that T_l is a cyclic group, we obtain from Lemma 5 that

$$\begin{aligned}\mathrm{ord}_l(U(r_\chi, k') : U(r_\chi p, k')) &\le 2^{i-1}[G : T_l \cap Z_p]_l(\mathrm{ord}_l|T_p|)^2 \qquad &&\text{if } p \ne l, \\ &\le 2[G : T_l]_l \mathrm{ord}_l|T_r| &&\text{if } p = l.\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{ord}_l(\varepsilon^- U_r : \varepsilon^- U_{rp}) &\le 2^{i-2}[G(k''/\mathbf{Q}) : Z(p, k'')][G : T_l \cap Z_p]_l(\mathrm{ord}_l|T_p|)^2 \quad &&\text{if } p \ne l, \\ &\le [G(k''/\mathbf{Q}) : Z(l, k'')][G : T_l]_l \mathrm{ord}_l|T_{m/l}| &&\text{if } p = l.\end{aligned}$$

These inequalities imply that

$$M_i \le \frac{2^{i-2}}{|P|} \sum_{p \in P \setminus \{l\}} [G : Z_p][Z_p : T_l \cap Z_p]_l(\mathrm{ord}_l|T_p|)^2 + \frac{\kappa}{|P|}[G : Z_l][Z_l : T_l]_l.$$

In view of (7), we then get the second inequality stated in the proposition.

PROPOSITION 3. *Suppose that $l = 2$ and that $(\varepsilon^- R : \varepsilon^- U)$ is even. Let P' be the set of prime numbers ramified in k' and let H be any cyclic subgroup of G. Then*

$$|P'| \ge 2,$$

$$\mathrm{ord}_2(\varepsilon^- R : \varepsilon^- U) \le \frac{2^{|P'|} - 2}{|P'| - \eta} \sum_{p \in P' \setminus \{2\}} (\mathrm{ord}_2|T_p|)^2 [G : Z_p][Z_p : H \cap Z_p]_2,$$

where $\eta = 1$ or 0 according as 2 lies in P' or not.

PROOF. As in the last section, let $\varepsilon^-_{k'} = \frac{1}{2}(1 - j|k')$, let f' be the product of primes in P' and, for each $\chi \in \mathfrak{X}_{k''}$, let f'_χ denote the product of prime divisors p of f' with $\chi(p) = 1$. By Theorem 5.2 of [**12**],

$$\begin{aligned}\mathrm{ord}_2(\varepsilon^- R : \varepsilon^- U) &= \sum_{\chi \in \mathfrak{X}_{k''}} \mathrm{ord}_2(\varepsilon^-_{k'} R(k') : \varepsilon^-_{k'} U(f'_\chi, k')) \\ &= \mathrm{ord}_2(\varepsilon^- R : \varepsilon^- U_{f'}).\end{aligned}$$

Since $2|(\varepsilon^- R : \varepsilon^- U)$, it then follows from Proposition 5.2 of [**12**] that $|P'| \ge 2$.

Now, to prove the second inequality of the proposition, we first consider the case where $2 \notin P'$. Let

$$M_i = \frac{1}{|P'|_{|P'|-1}C_{i-1}} \sum_{r,p} \mathrm{ord}_2(\varepsilon^- U_r : \varepsilon^- U_{rp}), \quad \text{for } i = 1, \ldots, |P'|;$$

where r ranges over the positive divisors of f' divisible by exactly $i-1$ prime numbers, with p ranging over all prime divisors of f'/r. Then, as in the proof of Proposition 2, we obtain from Theorem 5.2 of [**12**] and Lemma 5 that

$$\begin{aligned}\mathrm{ord}_2(\varepsilon^- U_r : \varepsilon^- U_{rp}) &= \sum_{\substack{\chi\in\mathfrak{X}_{k''}\\ \chi(p)=1}} \mathrm{ord}_2(\varepsilon_{k'}^- U(r_\chi, k') : \varepsilon_{k'}^- U(r_\chi p, k'))\\ &\leq 2^{i-1}[G(k''/\mathbf{Q}) : Z(p,k'')][G : H\cap Z_p]_2(\mathrm{ord}_2|T_p|)^2,\end{aligned}$$

so that

$$M_i \leq \frac{2^{i-1}}{|P'|}\sum_{p\in P'}[G : Z_p][Z_p : H\cap Z_p]_2(\mathrm{ord}_2|T_p|)^2, \quad \text{for each } i=1,\dots,|P'|.$$

Furthermore

$$\mathrm{ord}_2(\varepsilon^- R : \varepsilon^- U_{f'}) = \sum_{i=1}^{|P'|} M_i$$

and, by Proposition 5.2 of [**12**], $M_1 = 0$. Hence

$$\mathrm{ord}_2(\varepsilon^- R : \varepsilon^- U) = \sum_{i=2}^{|P'|} M_i \leq \frac{2^{|P'|}-2}{|P'|}\sum_{p\in P'}(\mathrm{ord}_2|T_p|)^2[G : Z_p][Z_p : H\cap Z_p]_2.$$

In the case $2\in P'$, we let

$$M_i = \frac{1}{(|P'|-1)_{|P'|-2}C_{i-1}}\sum_{r,p}\mathrm{ord}_2(\varepsilon^- U_{2r} : \varepsilon^- U_{2pr}), \quad \text{for } i=1,\dots,|P'|-1,$$

where r ranges over the positive integers dividing $f'/2$ and divisible by exactly $i-1$ prime numbers, with p ranging over all prime divisors of $f'/2r$. A similar argument as above enables us to see that

$$\begin{aligned}\mathrm{ord}_2(\varepsilon^- R : \varepsilon^- U) &= \sum_{i=1}^{|P'|-1} M_i\\ &\leq \frac{2^{|P'|}-2}{|P'|-1}\sum_{p\in P'\setminus\{2\}}(\mathrm{ord}_2|T_p|)^2[G : Z_p][Z_p : H\cap Z_p]_2.\end{aligned}$$

Thus we have proved the proposition.

4. This section is devoted to the proof of our main results, in which some formulas by Kida [**8**, **9**] will play an important part. For any abelian field K, as well as for k, we let K_∞ denote the basic $\mathbf{Z}_l$-extension over K. We also let $\lambda^- = \lambda^-(k)$.

THEOREM 1. *Suppose that $l>2$ and that $(\varepsilon^- R : \varepsilon^- U)$ is divisible by l. Then*

$$\lambda^- \geq 2(l-1), \quad \mathrm{ord}_l(\varepsilon^- R : \varepsilon^- U) \leq \frac{l(\lambda^- -1)}{2l-3}(8^{(\lambda^- -1)/(2l-3)} - 4).$$

PROOF. Let the notations P and κ be the same as in Proposition 2. Note that, by this proposition and the assumption $l|(\varepsilon^- R : \varepsilon^- U)$, $|P|\geq 3$ and so

$$[k_\infty : \mathbf{Q}_\infty]_l \geq l. \tag{8}$$

Now we apply the theorem of [8] (combined with [2]) to the extension k_∞/k''_∞:

$$\lambda^- - \delta = [k_\infty : \mathbb{Q}_\infty]_l(\lambda^-_{k''} - \delta) + \frac{1}{2}\sum_{p\in P\setminus\{l\}}(|T_p|_l - 1)[G(k_\infty/\mathbb{Q}) : Z(p,k_\infty)],$$

with $\delta = 1$ or 0 according as k contains a primitive pth root of unity or not. Let here

$$\Theta = \lambda^- + ([k_\infty : \mathbb{Q}_\infty]_l - 1)\delta$$

so that

$$\Theta = [k_\infty : \mathbb{Q}_\infty]_l\lambda^-_{k''} + \frac{1}{2}\sum_{p\in P\setminus\{l\}}(|T_p|_l - 1)[G(k_\infty/\mathbb{Q}) : Z(p,k_\infty)]. \tag{9}$$

We can see easily that

$$(l-1)(\mathrm{ord}_l|T_p|)^2 \le |T_p|_l - 1, \quad [G : Z_p] \le [G(k_\infty/\mathbb{Q}) : Z(p,k_\infty)],$$
$$[Z_p : T_l \cap Z_p]_l \le [k_\infty : \mathbb{Q}_\infty]_l, \quad \text{for } p \in P\setminus\{l\},$$
$$(l-1)\kappa \le [k_\infty : \mathbb{Q}_\infty]_l, \quad [G : Z_l][Z_l : T_l]_l \le [G(k''_\infty/\mathbb{Q}) : Z(l,k''_\infty)][k_\infty : \mathbb{Q}_\infty]_l$$

and from [7] that

$$\lambda^-_{k''} \ge \tfrac{1}{2}[G(k''_\infty/\mathbb{Q}) : Z(l,k''_\infty)], \quad \text{if } l \in P. \tag{10}$$

Therefore

$$\frac{l-1}{2}\left(\sum_{p\in P\setminus\{l\}}(\mathrm{ord}_l|T_p|)^2[G : Z_p][Z_p : T_l\cap Z_p]_l + \kappa[G : Z_l][Z_l : T_l]_l\right)$$
$$\le \Theta[k_\infty : \mathbb{Q}_\infty]_l.$$

Proposition 2 then implies that

$$\mathrm{ord}_l(\varepsilon^- R : \varepsilon^- U) \le \frac{2^{|P|} - 4}{(l-1)|P|}\Theta[k_\infty : \mathbb{Q}_\infty]_l. \tag{11}$$

We also find

$$\Theta \ge (l-1)|P| \tag{12}$$

by (8), (9), and (10).

Let us next prove

$$l(\lambda^- - 1)/(2l-3) \ge [k_\infty : \mathbb{Q}_\infty]_l. \tag{13}$$

Since $l > 2$ and $l|(\varepsilon^- R : \varepsilon^- U)$, it follows from Proposition 5.2, Theorem 5.2 of [12] that there exist a character χ in $\mathfrak{X}^-_{k''}$ and three distinct prime numbers p_1, p_2, p_3 in P with $\chi(p_1) = \chi(p_2) = \chi(p_3) = 1$. We put $W = \{p_1,p_2,p_3\}\setminus\{l\}$. Let ψ be a primitive Dirichlet character of degree l whose conductor is the product of primes in W. Then, by the theorem of [8],

$$\lambda^-(\mathbb{K}_\chi\mathbb{K}_\psi) - 1 \ge l(\lambda^-(\mathbb{K}_\chi) - 1) + (|W|/2)[\mathbb{K}_\chi : \mathbb{Q}](l-1)$$

since the primes in W are ramified in $\mathbb{K}_\psi$, unramified in $\mathbb{K}_\chi\mathbb{Q}_\infty$, and completely decomposed in $\mathbb{K}_\chi$. Note that if $l \in \{p_1,p_2,p_3\}$, then $\lambda^-(\mathbb{K}_\chi) \ge [\mathbb{K}_\chi : \mathbb{Q}]/2$ (cf. [7]). Hence

$$\lambda^-(\mathbb{K}_\psi k'') - 1 \ge \lambda^-(\mathbb{K}_\psi\mathbb{K}_\chi) - 1 \ge -l + \tfrac{3}{2}[\mathbb{K}_\chi : \mathbb{Q}](l-1) \ge 2l - 3,$$

so that

$$\lambda^-(\mathbf{K}_\psi k'') \geq 2(l-1).$$

On the other hand, again by the theorem of [**8**],

$$\begin{aligned}\lambda^-(\mathbf{K}_\psi k) - \delta &\geq [\mathbf{K}_\psi k_\infty : \mathbf{K}_\psi k''_\infty](\lambda^-(\mathbf{K}_\psi k'') - \delta)\\ &= (1/l)[\mathbf{K}_\psi k_\infty : \mathbf{Q}_\infty]_l(\lambda^-(\mathbf{K}_\psi k'') - \delta).\end{aligned}$$

Therefore

$$\lambda^-(\mathbf{K}_\psi k) - \delta \geq \frac{2l-2-\delta}{l}[\mathbf{K}_\psi k_\infty : \mathbf{Q}_\infty]_l.$$

In the case $\mathbf{K}_\psi \subseteq k_\infty$, this implies (13) by (8). In the case where $\mathbf{K}_\psi \not\subseteq k_\infty$, namely, $\mathbf{K}_\psi \cap k_\infty = \mathbf{Q}$, it follows that

$$\lambda^-(\mathbf{K}_\psi k) - \delta \geq (2l-2-\delta)[k_\infty : \mathbf{Q}_\infty]_l;$$

furthermore $\mathbf{K}_\psi k$ is unramified over k since the primes in W are ramified in k'. The theorem of [**8**] then shows that

$$\lambda^-(\mathbf{K}_\psi k) - \delta = l(\lambda^- - \delta)$$

and consequently that

$$\lambda^- - \delta \geq \frac{2l-2-\delta}{l}[k_\infty : \mathbf{Q}_\infty]_l.$$

Thus (13) is proved also when $\mathbf{K}_\psi \not\subseteq k_\infty$.

Now, by means of (13), we obtain from (8) that $\lambda^- \geq 2(l-1)$ and from the definition of Θ that

$$\Theta \leq \lambda^- - 1 + [k_\infty : \mathbf{Q}_\infty]_l \leq 3(l-1)(\lambda^- - 1)/(2l-3). \tag{14}$$

Furthermore

$$\frac{2^{|P|}-4}{|P|} \leq \frac{2^\omega - 4}{\omega}, \quad \text{with } \omega = \frac{3(\lambda^- - 1)}{2l-3} \tag{15}$$

since $|P| \leq \omega$ by (12) and (14). Finally, combining (11) with (13), (14), (15), we have

$$\mathrm{ord}_l(\varepsilon^- R : \varepsilon^- U) \leq (l\omega/3)(2^\omega - 4),$$

which was to be proved.

By Theorem 1, we get an upper bound for $a_l = \lim_{n\to\infty} \mathrm{ord}_l(\varepsilon_n^- R(k_n) : \varepsilon_n^- U(k_n))$, $l > 2$, depending only on λ^-.

COROLLARY 1. *If $l > 2$, then*

$$a_l \leq \frac{l(\lambda^- - 1)}{2l-3}(8^{(\lambda^- - 1)/(2l-3)} - 4).$$

PROOF. We may assume $a_l > 0$. This and Theorem 6.1 of [**12**] imply that for every sufficiently large integer $n \geq 0$,

$$l | (\varepsilon_n^- R(k_n) : \varepsilon_n^- U(k_n)) \quad \text{and} \quad a_l = \mathrm{ord}_l(\varepsilon_n^- R(k_n) : \varepsilon_n^- U(k_n)).$$

We thus obtain the corollary, replacing k by k_n in Theorem 1.

THEOREM 2. *Suppose that $l = 2$. Then*

$$\operatorname{ord}_2(\varepsilon^- R : \varepsilon^- U) \leq \tfrac{64}{3}(5\lambda^- + 4)(4^{\lambda^- + 1} - 1).$$

PROOF. We may also suppose that $2 | (\varepsilon^- R : \varepsilon^- U)$. Let P^* be the set of odd primes ramified in k' and let g be the number of such primes; $g = |P^*|$. By Proposition 3, we have $g \geq 1$ and, for any cyclic subgroup H of G,

$$\operatorname{ord}_2(\varepsilon^- R : \varepsilon^- U) \leq \frac{2^{g+1} - 2}{g} \sum_{p \in P^*} (\operatorname{ord}_2 |T_p|)^2 [G : Z_p][G : H]_2.$$

Hence it follows that

$$(16) \quad \operatorname{ord}_2(\varepsilon^- R : \varepsilon^- U) \leq \frac{2^{g+1} - 2}{g} \left(\frac{4}{3} \sum_{p \in P^*} (|T_p|_2 - 1)[G(k_\infty/\mathbf{Q}) : Z(p, k_\infty)] \right) [k_\infty : \mathbf{Q}_\infty]_2.$$

Let P^- be the set of primes in P^* whose prime divisors of k^+ are ramified in k. Note that a prime of k_∞^+ lying above some prime in $P^* \backslash P^-$ is always decomposed in k_∞. Let c be the number of finite primes of k_∞^+ which are ramified in k_∞ and not lying above 2;

$$c = \sum_{p \in P^-} [G(k_\infty/\mathbf{Q}) : Z(p, k_\infty)].$$

Then Theorems 1, 2 of [**9**], together with [**1**], show that

$$\lambda^- \geq -[k_\infty^+ : k_\infty''] + \sum_{\mathfrak{B}} (e_{\mathfrak{B}} - 1) + c,$$

where $\mathfrak{B}$ ranges over the finite primes of k_∞^+ not lying above 2 and, for each such prime $\mathfrak{B}$, $e_{\mathfrak{B}}$ denotes the ramification index of $\mathfrak{B}$ for k_∞^+/k_∞''. Since

$$\begin{aligned}\sum_{\mathfrak{B}} (e_{\mathfrak{B}} - 1) = &\sum_{p \in P^-} \left(\frac{1}{2} |T_p|_2 - 1 \right) [G(k_\infty/\mathbf{Q}) : Z(p, k_\infty)] \\ &+ \frac{1}{2} \sum_{p \in P^* \backslash P^-} (|T_p|_2 - 1)[G(k_\infty/\mathbf{Q}) : Z(p, k_\infty)],\end{aligned}$$

we obtain from the above that

$$\lambda^- + \frac{1}{2}[k_\infty : \mathbf{Q}_\infty]_2 \geq \frac{1}{2} \sum_{p \in P^*} (|T_p|_2 - 1)[G(k_\infty/\mathbf{Q}) : Z(p, k_\infty)].$$

Therefore, by (16),

$$(17) \qquad \operatorname{ord}_2(\varepsilon^- R : \varepsilon^- U) \leq \frac{16(2^g - 1)}{3g} (\lambda^- + \tfrac{1}{2}[k_\infty : \mathbf{Q}_\infty]_2)[k_\infty : \mathbf{Q}_\infty]_2.$$

Now we put

$$\Delta = -1 + \sum_{p \in P^-} N_p + \sum_{p \in P^* \backslash P^-} N_p \left(1 - \frac{1}{|T_p|_2} \right),$$

with $N_p = [G(\mathbf{Q}_\infty/\mathbf{Q}) : Z(p, \mathbf{Q}_\infty)]$ for each $p \in P^*$. Then, again by **[9]** (and **[1]**),

$$\lambda^- \geq \lambda^-(k') \geq \tfrac{1}{2}[k'_\infty : \mathbf{Q}_\infty]\Delta.$$

Here, as $[k'_\infty : \mathbf{Q}_\infty] \geq 2$ and $\Delta \geq g/2 - 1$, it follows that

$$g \leq 2(\lambda^- + 1). \tag{18}$$

Furthermore each $|T_p|_2$ divides $4N_p$, the highest power of 2 dividing $(p-1)(p+1)/2$, so that 4Δ is an integer ≥ -2. Hence it also follows that

$$\lambda^- \geq \tfrac{1}{8}[k_\infty : \mathbf{Q}_\infty]_2, \quad \text{if } \Delta > 0.$$

On the other hand, if $\Delta \leq 0$, namely, if $\Delta = -\frac{1}{2}, -\frac{1}{4}$ or 0, then $g = 1$ or 2 and it is easy to see that

$$[k_\infty : \mathbf{Q}_\infty]_2 \leq 2 \prod_{p \in P^*} |T_p|_2 \leq 8.$$

Consequently

$$[k_\infty : \mathbf{Q}_\infty]_2 \leq 8(\lambda^- + 1).$$

The theorem now follows from this, (17), and (18).

COROLLARY 2. *In the case* $l = 2$,

$$a_2 \leq \tfrac{64}{3}(5\lambda^- + 4)(4^{\lambda^- + 1} - 1).$$

By Theorems 1 and 2, we also have the following result.

COROLLARY 3.

$$\begin{aligned} \lambda^- &\geq \frac{2l-3}{e^{-1} + \log 8} \log\left(\frac{1}{l}\mathrm{ord}_l(\varepsilon^- R : \varepsilon^- U) + 4\right) + 1 \quad && \textit{if } l > 2,\ l \mid (\varepsilon^- R : \varepsilon^- U); \\ &\geq \frac{\log(\frac{3}{320}\mathrm{ord}_2(\varepsilon^- R : \varepsilon^- U) + \frac{8}{5})}{e^{-1} + \log 4} - 1 && \textit{if } l = 2. \end{aligned}$$

In particular, $\lambda^- \to \infty$ *as* k *varies through a sequence of imaginary abelian fields such that* $\mathrm{ord}_l(\varepsilon^- R : \varepsilon^- U) \to \infty$.

PROOF. In the case $l > 2$, we put $\xi = (\lambda^- - 1)/(2l-3)$, assuming $l \mid (\varepsilon^- R : \varepsilon^- U)$. Theorem 1 then implies that $\xi \geq 1$ and

$$(1/l)\mathrm{ord}_l(\varepsilon^- R : \varepsilon^- U) \leq \xi(8^\xi - 4) \leq \xi 8^\xi - 4 \leq e^{\xi/e + \xi \log 8} - 4.$$

Thus we obtain the first inequality of the corollary. In the case $l = 2$, it follows from Theorem 2 that

$$\begin{aligned} \tfrac{3}{64}\mathrm{ord}_2(\varepsilon^- R : \varepsilon^- U) &\leq (5\lambda^- + 4)(4^{\lambda^- + 1} - 1) \leq 5(\lambda^- + 1)4^{\lambda^- + 1} - 8 \\ &\leq 5e^{(e^{-1} + \log 4)(\lambda^- + 1)} - 8; \end{aligned}$$

and consequently the second inequality is proved.

5. We conclude the paper with a few additional results. Let, for simplicity,

$$b(\lambda^-) = \tfrac{64}{3}(5\lambda^- + 4)(4^{\lambda^-+1} - 1).$$

PROPOSITION 4. *Assume that $l = 2$. Then the following assertions hold:*
(i) *for any element α of $\mathfrak{S}$,* $\mathrm{ord}_2(\alpha U_2 : \alpha U) \le b(\lambda^-)$,
(ii) $\mathrm{ord}_2(R : U) \le b(\lambda^-)$,
(iii) $\mathrm{ord}_2(\varepsilon^+ R : \varepsilon^+ U) \le \iota 2^{[G\,:\,Z_2]} + b(\lambda^-)$,
where $\iota = 1$ or 0 according as $j|k$ lies in T_2 or not.

PROOF. The assertion (i) can be shown in almost the same way as Theorem 2. Note that $U_2 = R$ if 2 is unramified in k. The assertions (ii), (iii) follow from (i) and from Proposition 5.2 of [**12**].

REMARK. The "generalized" index $(\varepsilon^+ R : \varepsilon^+ U)$ is considered the main part of the "supplementary factor" of an algebraic class number formula for k^+ (cf. Theorem 4.1 of [**12**]).

Next, we see below that a similar result as Proposition 4 does not hold for the case $l > 2$.

PROPOSITION 5. *Assume that $l > 2$. Then there exists a tower of imaginary abelian fields $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \subseteq \cdots$ such that*

$$\begin{aligned} &\lambda^-(K_n) = 0 && \textit{for every } n \ge 1, \\ &\mathrm{ord}_l(R(K_n) : U(K_n)) \to \infty && \textit{as } n \to \infty, \end{aligned}$$

and for any prime number p,

$$\mathrm{ord}_l(\varepsilon_n^+ U(p, K_n) : \varepsilon_n^+ U(K_n)) \to \infty \quad \textit{as } n \to \infty,$$

with $\varepsilon_n^+ = \frac{1}{2}(1 + j|K_n)$, $n \ge 1$.

PROOF. Let us first take an imaginary quadratic field K with $\lambda^-(K) = 0$ which is not contained in the cyclotomic field of lth roots of unity (for the infinite existence of such fields, see [**5**]). It follows that there exist infinitely many prime numbers $\equiv 1 \pmod{l}$ undecomposed in K. Let $p_1 < p_2 < \cdots < p_n < \cdots$ be the increasing sequence of such primes and let F_n denote, for each $n \ge 1$, the cyclic field of degree l with conductor p_n. Let L be the compositum of F_n for all $n \ge 1$, so that L is an infinite abelian extension over $\mathbf{Q}$, with $G(L/\mathbf{Q})$ isomorphic to the direct product of $G(F_n/\mathbf{Q})$, $n \ge 1$. We next take, for each $n \ge 1$, an automorphism s_n in $G(L/\mathbf{Q})$ such that the restriction $s_n|F_n$ is a generator of the cyclic group $G(F_n/\mathbf{Q})$ and such that $s_n|F_m$ is the identity map in $G(F_m/\mathbf{Q})$ for every positive integer $m \ne n$. Furthermore let L_n, $n \ge 1$, be the abelian field consisting of all elements in the compositum $F_1 \cdots F_n$ fixed by $s_1 \cdots s_n$. We then put

$$K_n = KL_n \quad \text{for each } n \ge 1.$$

Note that, since $p_1, \ldots, p_n$ are undecomposed in K, the primes of $L_n\mathbf{Q}_\infty = K_n^+\mathbf{Q}_\infty$ ramified for $L_n\mathbf{Q}_\infty/\mathbf{Q}_\infty$ are undecomposed in $K_n\mathbf{Q}_\infty$. It is now easy to see from the theorem of [**8**] that

$$\lambda^-(K_n) = 0, \qquad n \ge 1.$$

Letting $\varepsilon_n^+ = \frac{1}{2}(1+j|K_n)$ in $\mathfrak{S}(K_n)$ for each $n \geq 1$, we also see from Theorems 5.2, 5.4 of [**12**] that

$$\begin{aligned}\operatorname{ord}_l(\varepsilon_n^+ R(K_n) : \varepsilon_n^+(K_n)) &= \operatorname{ord}_l(R(K_n) : U(K_n)) \\ &= \operatorname{ord}_l(R(L_n) : U(L_n)) = 2^{n-2} - 1, \qquad n \geq 2,\end{aligned}$$

and that, for any prime number p,

$$\operatorname{ord}_l(\varepsilon_n^+ R(K_n) : \varepsilon_n^+ U(p, K_n)) = 0, \qquad n \geq 1.$$

Of course $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \subseteq \cdots$, and this completes the proof of Proposition 5.

REFERENCES

1. B. Ferrero, *Iwasawa invariants of abelian number fields*, Math. Ann. **234** (1978), 9–24.
2. B. Ferrero and L. C. Washington, *The Iwasawa invariant μ_p vanishes for abelian number fields*, Ann. of Math. (2) **109** (1979), 377–395.
3. H. Hasse, *Über die Klassenzahl abelscher Zahlkörper*, Akademie-Verlag, Berlin, 1952.
4. K. Horie, *On the index of the Stickelberger ideal and the cyclotomic regulator*, J. Number Theory **20** (1985), 238–253.
5. ——, *A note on basic Iwasawa λ-invariants of imaginary quadratic fields*, Invent. Math. **88** (1987), 31–38.
6. K. Iwasawa, *A class number formula for cyclotomic fields*, Ann. of Math. (2) **76** (1962), 171–179.
7. ——, *On $\mathbf{Z}_l$-extensions of algebraic number fields*, Ann. of Math. (2) **98** (1973), 246–326.
8. Y. Kida, *l-extensions of CM-fields and cyclotomic invariants*, J. Number Theory **12** (1980), 519–528.
9. ——, *Cyclotomic $\mathbf{Z}_2$-extensions of J-fields*, J. Number Theory **14** (1982), 353–361.
10. T. Kimura and K. Horie, *On the Stickelberger ideal and the relative class number*, Trans. Amer. Math. Soc. **302** (1987), 727–739.
11. W. Sinnott, *On the Stickelberger ideal and the circular units of a cyclotomic field*, Ann. of Math. (2) **108** (1978), 107–134.
12. ——, *On the Stickelberger ideal and the circular units of an abelian field*, Invent. Math. **62** (1980), 181–234.
13. B. L. van der Waerden, *Moderne Algebra*. II, Springer-Verlag, Berlin, 1940.

DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY, SETAGAYA-KU, TOKYO 158, JAPAN

Current address: Department of Mathematics, Yamaguchi University, Yoshida, Yamaguchi 753, Japan